2010

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State what it means for a real sequence to converge.

We say that the sequence $\langle x_n \rangle$ converges to l (and write $\lim x_n = l$) if, given $\epsilon > 0$ we can find a number N, such that, whenever $n > N$, we have $|x_n - l| < \epsilon$.

(or $\forall \epsilon > 0$, $\exists N$ such that $n > N \Longrightarrow |x_n - l| < \epsilon$.)

(b) Use the definition of convergence (not the combination theorem and other theorems) to prove that

$$\lim_{n\to\infty} \frac{5 \cdot 2^n - 4}{2^n - 1} = 5.$$

We have

$$\frac{5 \cdot 2^n - 4}{2^n - 1} - 5 = \frac{5 \cdot 2^n - 4}{2^n - 2} - \frac{5(2^n - 1)}{2^n - 1} = \frac{5 \cdot 2^n - 4 - 5 \cdot 2^n + 5}{2^n - 1} = \frac{1}{2^n - 1}.$$

To make $|x_n - 5| < \epsilon$, we need to have

$$\left|\frac{1}{2^n - 1}\right| < \epsilon \Leftrightarrow \frac{1}{2^n - 1} < \epsilon \Leftrightarrow 2^n - 1 > \frac{1}{\epsilon}$$

$$\Leftrightarrow 2^n > \frac{1}{\epsilon} + 1. \tag{1}$$

It suffices to have $2^n > \frac{1}{\epsilon} + 1$. Since $2^n > n$, it suffices to have

$$n > \frac{1}{\epsilon} + 1. \tag{2}$$

So we take $N = 1/\epsilon + 1$. Then $n > N$ implies (2) and this implies (1), which is equivalent to $|x_n - 5| < \epsilon$.

(c) State the Least Upper Bound principle (continuum property)

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Every non-empty set which is bounded above has a least upper bound.

(d) Let $\langle x_n \rangle$ be an increasing sequence which is bounded above. Show that it converges to its smallest upper bound (supremum).

Let $S = \{x_n, n \in \mathbb{N}\}$. Since the sequence $\langle x_n \rangle$ is bounded above, the set S is bounded above and (since it is non-empty) the least upper bound property guarantees the existence of the smallest upper bound of S, which we call, say, B. We need to prove

$$\lim_{n\to\infty} x_n = B.$$

This is equivalent, by the definition of convergence to

$$\forall \epsilon > 0 \quad \exists N : n > N \Longrightarrow |x_n - B| < \epsilon.$$

Since $x_n \leq B$, as B is an upper bound for the sequence, we have

$$|x_n - B| = B - x_n.$$

Given $\epsilon > 0$, we know that $B - \epsilon$ is not an upper bound for the sequence, as its smallest upper bound is B and $B - \epsilon < B$. This means that we can find a subscript N, such that

$$x_N > B - \epsilon.$$

Since $\langle x_n \rangle$ is an increasing sequence, we have

$$n > N \Longrightarrow x_n \geq x_N.$$

Moreover, $x_n \leq B$ for all $n \in \mathbb{N}$. Therefore,

$$n > N \Longrightarrow B - \epsilon < x_N \leq x_n \leq B \Longrightarrow B - \epsilon < x_n \leq B \Longrightarrow |x_n - B| = B - x_n < \epsilon.$$

2. (a) State the definition of $\lim_{x\to\xi} f(x) = l$.

We say that the limit of $f(x)$ as x tends to ξ is l, if, given $\epsilon > 0$ we can find a $\delta > 0$, such that: whenever $0 < |x - \xi| < \delta$, we have $|f(x) - l| < \epsilon$.

(or $\forall \epsilon > 0$, $\exists \delta > 0$ such that $0 < |x - \xi| < \delta \Longrightarrow |f(x) - l| < \epsilon$.)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & (x < 2), \\ 5, & (x \geq 2). \end{cases}$$

(i) Prove carefully (using ϵ and δ) that $f(x)$ is continuous at $\xi = 0$.

(ii) Compute carefully (using ϵ and δ) the limits

$$\lim_{x\to 2-} f(x), \quad \lim_{x\to 2+} f(x).$$

Is $f(x)$ continuous at 2?

(i) We need to prove:

$$\forall\epsilon > 0, \quad \exists\delta > 0 : |x| < \delta \Longrightarrow |x^2 - 0| < \epsilon.$$

Since $|x^2| = |x|^2$ the condition $|x^2| < \epsilon$ is equivalent to $|x| < \sqrt{\epsilon}$. We take $\delta = \sqrt{\epsilon} > 0$.

(ii) We will prove that

$$\lim_{x\to 2-} f(x) = 4, \quad \lim_{x\to 2+} f(x) = 5.$$

Since the limits are not equal, the limit $\lim_{x\to 2} f(x)$ does not exist and the function is not continuous at 2.

For $\lim_{x\to 2-} f(x) = 4$ we need to prove

$$\forall\epsilon > 0 \quad \exists\delta > 0 : 2 - \delta < x < 2 \Longrightarrow |f(x) - 4| < \epsilon.$$

For $x < 2$ we have $f(x) = x^2 < 4$. Therefore, $|f(x) - 4| = 4 - x^2$. This gives

$$|f(x) - 4| < \epsilon \Leftrightarrow 4 - x^2 < \epsilon \Leftrightarrow 4 - \epsilon < x^2.$$

If $\epsilon \leq 4$ we get

$$|f(x) - 4| < \epsilon \Leftrightarrow \sqrt{4-\epsilon} < |x|.$$

We can restrict our attention to $x > 0$ (and < 2). Then we get

$$|f(x) - 4| < \epsilon \Leftrightarrow \sqrt{4-\epsilon} < x.$$

We match

$$2 - \delta = \sqrt{4-\epsilon} \Leftrightarrow \delta = 2 - \sqrt{4-\epsilon}.$$

We only need to prove that

$$\delta > 0 \Leftrightarrow 2 - \sqrt{4-\epsilon} > 0 \Leftrightarrow 2 > \sqrt{4-\epsilon} \Leftrightarrow 4 > 4 - \epsilon \Leftrightarrow 0 > -\epsilon,$$

which is true. The case $\epsilon > 4$ is even easier: In this case $4 - x^2 < \epsilon$ is always true ($x^2 \geq 0$) as

$$4 - x^2 \leq 4 < \epsilon.$$

This means we can take any $\delta > 0$ with $\delta < 2$ in this case. The condition $\delta < 2$ guarantees we are still working with $x > 0$.

For $\lim_{x\to 2+} f(x) = 5$ we need to prove

$$\forall\epsilon > 0 \quad \exists\delta > 0 : 2 < x < 2 + \delta \Longrightarrow |f(x) - 5| < \epsilon.$$

But for $x > 2$ we have $f(x) = 5$, therefore,

$$|f(x) - 5| = |5 - 5| = 0 < \epsilon$$

and this holds for whatever $\delta > 0$ we take. So we have proved the right-hand limit.

(c) Let f be continuous on the compact interval $[a, b]$. We have proved (and you can assume) that f is bounded on $[a, b]$.

Prove that f achieves a maximum value d and a minimum value c on $[a, b]$.

Method 1: We know that $f(x)$ is bounded above. By the continuum property of the real numbers,

$$S = \{f(x); x \in [a, b]\}$$

has a supremum, call it d. We must find a $\xi \in [a, b]$ with $f(\xi) = d$. Assume that this is not true, i.e. for all $x \in [a, b]$ we have $f(x) \neq d$. Define the function

$$g : [a, b] \to \mathbb{R}, \quad g(x) = \frac{1}{d - f(x)}.$$

Since $f(x) \neq 0$ and $f(x)$ is continuous on $[a, b]$, the combination theorem implies that $g(x)$ is also continuous on $[a, b]$. Since $f(x) < d$, we have $g(x) > 0$. By the theorem that says:

Theorem: Let f be continuous on the compact interval $[a, b]$. Then f is bounded on $[a, b]$.

we know that $g(x)$ is bounded above on the interval $[a, b]$, say by M, which has to be positive. Therefore,

$$g(x) \leq M \Leftrightarrow \frac{1}{d - f(x)} \leq M \Leftrightarrow d - f(x) \geq \frac{1}{M} \Leftrightarrow d - \frac{1}{M} \geq f(x). \quad \forall x \in [a, b].$$

So $d - 1/M$ is an upper bound of S, less than the smallest upper bound d (the supremum). This is a contradiction. Therefore, there exists a $\xi \in [a, b]$ with $f(\xi) = d$.

For the minimum we can use $-f(x)$, which is continuous on $[a, b]$, so it achieves its maximum on $[a, b]$ by the statement just proven. Say, this maximum is $-c$. Then for some $\xi \in [a, b]$ we have $-f(\xi) = -c \Longrightarrow f(\xi) = c$ and c is the minimum of $f(x)$ on $[a, b]$.

Method 2: We know that $f(x)$ is bounded above. By the continuum property of the real numbers,

$$S = \{f(x); x \in [a, b]\}$$

has a supremum, call it d. We must find a $\xi \in [a, b]$ with $f(\xi) = d$. Consider $d - 1/n < d$. Since d is the smallest upper bound of S, $d - 1/n$ is not an upper bound of S. So we can find $x_n \in [a, b]$ with $f(x_n) > d - 1/n$. This produces a sequence $\langle x_n \rangle$, $n = 1, 2, \ldots$, with $a \leq x_n \leq b$. This sequence is bounded, so it has a convergent subsequence x_{n_r}, $r = 1, 2, \ldots$ by the Bolzano–Weierstrass theorem. Call its limit ξ. As $a \leq x_{n_r} \leq b$, we also have $a \leq \xi = \lim x_{n_r} \leq b$. By the continuity of $f(x)$

$$f(\xi) = \lim f(x_{n_r}).$$

Now we have

$$d \geq f(x_{n_r}) > d - \frac{1}{n_r},$$

where $1/n_r \to 0$. By the sandwich theorem $d \geq f(\xi) \geq d$. So $f(\xi) = d$ as required.

For the minimum we can use $-f(x)$, which is continuous on $[a, b]$, so it achieves its maximum on $[a, b]$ by the statement just proven. Say, this maximum is $-c$. Then for some $\xi \in [a, b]$ we have $-f(\xi) = -c \Longrightarrow f(\xi) = c$ and c is the minimum of $f(x)$ on $[a, b]$.

(d) Can you apply the theorems in (c) to the function $f(x)$ in (b) on the interval $[0, 3]$? Determine (with explanations) whether the function $f(x)$ is bounded and/or achieves its maximum value on $[0, 3]$.

We cannot apply the theorems above, as the function is not continuous on the interval $[0, 3]$, because it is not continuous at $\xi = 2$. This does not mean that the conclusion of the theorems is false. In fact, $f(x)$ is bounded and achieves a maximum on $[0, 3]$:

For $0 \leq x < 2$, $f(x) = x^2 \in [0, 4)$. For $x \geq 2$, $f(x) = 5 \leq 5$. The upper bound of $f(x)$ is 5 on the interval $[0, 5]$. In fact $f(x)$ achieves the maximum 5 at any point $x \in [2, 3]$. The function is also bounded below by 0.

3. (a) (i) Define e^x for $x \in \mathbb{R}$ as a series. Show that the series converges for all $x \in \mathbb{R}$.

Definition of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots.$$

We use the ratio test to show convergence for all $x \in \mathbb{R}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \left|\frac{x^{n+1}n!}{x^n(n+1)!}\right| = \left|\frac{x}{n+1}\right| = \frac{|x|}{n+1} \to 0,$$

as $n \to \infty$. Since $0 < 1$, the ratio tests gives convergence for all $x \in \mathbb{R}$.

(ii) Show that for $0 \leq x < 1$ we have

$$e^x \leq \frac{1}{1-x}.$$

We have for $x \geq 0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \leq 1 + x + x^2 + x^3 + x^4 + \cdots$$

as the denominators are larger than 1. If, moreover, $0 \leq x < 1$, the right-hand side is a geometric series which converges. Its sum is

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}.$$

The result follows.

(iii) Show that the series $\sum_{n=2}^{\infty}(1 - e^{-1/n})^2$ converges.

Hint: Use (ii) for a particular choice of x.

We take $x = 1/n < 1$ for $n \geq 2$. We get

$$e^{1/n} \leq \frac{1}{1-1/n} \Longrightarrow e^{-1/n} \geq 1 - \frac{1}{n} \Longrightarrow \frac{1}{n} \geq 1 - e^{-1/n} \Longrightarrow \frac{1}{n^2} \geq (1 - e^{-1/n})^2.$$

The series converges by the comparison test comparing with the larger series

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \zeta(2) - 1.$$

Second method: We know that

$$e^x \geq 1 + x, \quad \forall x \in \mathbb{R}.$$

(Actually for $-1 < x < 0$ this was proved using (ii)). We apply this with $x = -1/n$. We get

$$e^{-1/n} \geq 1 - \frac{1}{n} \Longrightarrow \frac{1}{n} \geq 1 - e^{-1/n}.$$

The rest of the proof continuous as above.

(b)

Let a_n be positive for all $n \in \mathbb{N}$. If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, show that the series

$$\sum_{n=1}^{\infty} \sqrt{a_n a_{2n}}$$

converges as well.

The AM–GM inequality gives:

$$\sqrt{a_n a_{2n}} \leq \frac{1}{2}(a_n + a_{2n}),$$

so we compare with the series

$$\sum_{n=1}^{\infty} \frac{1}{2}(a_n + a_{2n}) = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} a_{2n}\right). \tag{3}$$

The first series on the right is convergent by the assumption of the problem. Let s_n be the partial sums of the series $\sum_n a_n$. The sequence s_n is convergent and, therefore, bounded. Let t_n be the partial sums of the series $\sum_n a_{2n}$, i.e.

$$t_n = a_2 + a_4 + \cdots + a_{2n}.$$

Clearly, as all a_n are positive, we have $t_n < s_{2n}$. This implies that the increasing sequence t_n is bounded above. So it converges. By definition, this means the corresponding series $\sum_n a_{2n}$ is convergent. The sum of the two convergent series on the right of (3) is convergent and so is 1/2 of these. By the comparison test the series $\sum_n \sqrt{a_n a_{2n}}$ is convergent.

(c) State the Cauchy–Schwarz inequality.

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{1/2} \left(b_1^2 + b_2^2 + \cdots + b_n^2\right)^{1/2},$$

or

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

4. (a) Define what it means for a sequence to be Cauchy.

We say that the sequence $\langle x_n \rangle$ is Cauchy, if, given $\epsilon > 0$ we can find a number N, such that, whenever $n > N$, and $m > N$, we have $|x_n - x_m| < \epsilon$.

(or $\forall \epsilon > 0$, $\exists N$ such that $n > N$ and $m > N \Longrightarrow |x_n - x_m| < \epsilon$.)

(b) State the General Principle of Convergence.

A sequence is convergent if and only if it is a Cauchy sequence.

(c) Prove that every convergent sequence is a Cauchy sequence.

We are given that the sequence has a limit, say l, i.e. $\lim_n x_n = l$. This means: Given $\epsilon > 0$ we can find N such that

$$n > N \Longrightarrow |x_n - l| < \frac{\epsilon}{2}.$$

We are asked to prove that it is Cauchy, i.e. $\forall \epsilon > 0$, $\exists N$ such that $n > N$ and $m > N \Longrightarrow |x_n - x_m| < \epsilon$.

So, given $\epsilon > 0$ we use the same N as in the equation above to deduce that, whenever $n > N$ and $m > N$ we have at the same time $|x_n - l| < \epsilon/2$ and $|x_m - l| < \epsilon/2$. Then

$$|x_n - x_m| = |x_n - l + l - x_m| \leq |x_n - l| + |l - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(d) Prove that every sequence has a monotone subsequence.

A point x_N is called a peak point for the sequence, if

$$n > N \Longrightarrow x_n \leq x_N.$$

There are two possibilities. Either the sequence has infinitely many peak points, or finitely many peak points.

Case 1: There are infinitely many peak points, i.e. we can find a sequence of integers $n_1, n_2, \ldots$, which is increasing ($n_1 < n_2 < \ldots$) such that the points $x_{n_1}, x_{n_2} \ldots$ are all peak points.

Then $x_{n_1} \geq x_{n_2}$, as x_{n_1} is a peak point. Also, $x_{n_2} \geq x_{n_3}$, as x_{n_2} is a peak point. We continue this way to get

$$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots$$

i.e. we produced a subsequence which is decreasing.

Case 2: The are only finitely many peak points. Then we can find a largest subscript N, such that x_N is the last peak point. Set $n_1 = N + 1$. Since x_{N+1} is not a peak point, we can find a subscript $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. Since x_{n_2} is not a peak point we can find a subscript $n_3 > n_2$ with $x_{n_3} > x_{n_2}$. We continue this way to produce an increasing subsequence

$$x_{n_1} < x_{n_2} < x_{n_3} < \cdots.$$

(e) State and prove the Bolzano–Weierstrass Theorem.

Theorem: (Bolzano–Weierstrass) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof: Let $\langle x_n \rangle$ be a bounded sequence. By (d) above, it has a monotone subsequence, say $\langle x_{n_r} \rangle$. Since the whole sequence $\langle x_n \rangle$ is bounded, the subsequence $\langle x_{n_r} \rangle$ is also bounded. So $\langle x_{n_r} \rangle$ is a monotone and bounded sequence. Such a sequence converges to its supremum (if it is increasing), see Problem 1 (d), or to its infimum (if it is decreasing).

5. (a) State and prove the Intermediate Value Theorem.

Theorem. Let $f : [a, b] \to \mathbf{R}$ be continuous. If λ lies between $f(a)$ and $f(b)$, then we can find a ξ between a and b such that $\lambda = f(\xi)$.

Proof: We are given that either $f(a) < \lambda < f(b)$ or $f(a) > \lambda > f(b)$. Assume that $f(a) < \lambda < f(b)$. Define

$$S = \{x \in [a, b], f(x) < \lambda\}.$$

Then $S \neq \emptyset$. This is so, because a belongs to it. This set is also bounded above by b. The continuum property implies that S has a least upper bound, call it ξ. Since $a \in S$ and b is an upper bound of S, we have $\xi \in [a, b]$. We will show that $f(\xi) = \lambda$. This will be done by contradiction. If $f(\xi) \neq \lambda$, then there are two cases: (i) $f(\xi) < \lambda$ (ii) $f(\xi) > \lambda$.

(i) Since $f(\xi) < \lambda$, $\epsilon = \lambda - f(\xi) > 0$. Since f is continuous at ξ, we can find a $\delta > 0$ such that

$$|x - \xi| < \delta \Longrightarrow |f(x) - f(\xi)| < \lambda - f(\xi) = \epsilon.$$

The inequality

$$f(x) - f(\xi) \leq |f(x) - f(\xi)| < \lambda - f(\xi) \Longrightarrow f(x) < \lambda$$

means that for $\xi - \delta < x < \xi + \delta$ we have $f(x) < \lambda$. Any $y \in (\xi, \xi + \delta)$ therefore belongs to S, but ξ is an upper bound of S. This is a contradiction.

(ii) Since $f(\xi) > \lambda$, $\epsilon = f(\xi) - \lambda > 0$. Since f is continuous at ξ, we can find a $\delta > 0$ such that

$$|x - \xi| < \delta \Longrightarrow |f(x) - f(\xi)| < f(\xi) - \lambda = \epsilon.$$

The inequality

$$f(\xi) - f(x) \leq |f(x) - f(\xi)| < f(\xi) - \lambda \Longrightarrow -f(x) < -\lambda \Longrightarrow f(x) > \lambda.$$

So on the whole interval $(\xi - \delta, \xi + \delta)$ the function has values larger than λ. Take any $z \in (\xi - \delta, \xi)$. We will show that it is an upper bound of S, while it is smaller that the least upper bound of S, namely ξ. This provides the desired contradiction.

Let $x \in S$. This means that $f(x) < \lambda$. Since on $(\xi - \delta, \xi + \delta)$ the function has values larger than λ, x has to be less than $\xi - \delta$. This means that $x < z$. So z is an upper bound of S as required.

If $f(b) < \lambda < f(a)$, then we consider $-f : [a, b] \to \mathbf{R}$. We apply the previous result to this function and $-\lambda$, since $-f(a) < -\lambda < -f(b)$. We find a ξ in $[a, b]$ with $-f(\xi) = -\lambda$. This suffices.

(b) Let $f : [0, 1] \to [0, 1]$ be continuous and let $g : [0, 1] \to \mathbb{R}$ be continuous with

$$g(0) = 0, \quad g(1) = 1.$$

Prove that we can find a $\xi \in [0, 1]$ with $f(\xi) = g(\xi)$.

We consider the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(x) = f(x) - g(x).$$

Since f and g are continuous, the combination theorem gives that h is continuous. Moreover,

$$h(0) = f(0) - g(0) = f(0) \geq 0, \quad h(1) = f(1) - g(1) = f(1) - 1 \leq 0,$$

by the given values of g and the fact that the range of f in contained in $[0,1]$.

Case 1: $f(0) = 0$, then we take $\xi = 0$.

Case 2: $f(1) = 1$, then we take $\xi = 1$.

Case 3: None of the above. This gives

$$h(0) > 0, \quad h(1) < 0.$$

We apply the Intermediate value theorem with $\lambda = 0$ to deduce that there exists a $\xi \in (0,1)$ with $h(\xi) = 0 = f(\xi) - g(\xi)$. This gives $f(\xi) = g(\xi)$.

(c) Let $f(x)$ be a continuous function on $[0,\infty)$ with $f(0) = 4$. We assume we can find a constant k such that

$$\ln|f(x)| = x + k, \quad \forall x \in [0,\infty).$$

Show that

$$f(x) = 4e^x, \quad \forall x \in [0,\infty).$$

We exponentiate $\ln|f(x)| = x + k$ to get

$$|f(x)| = e^k \cdot e^x \Longrightarrow f(x) = \pm e^k e^x.$$

We plug $f(0) = 4$ to get $4 = \pm e^k e^0 \Longrightarrow 4 = +e^k$. This gives

$$f(x) = \pm 4e^x.$$

We need to prove that $+$ is the correct sign for all $x \in [0,\infty)$. It certainly is correct for $x = 0$, as $f(0) = 4 = 4e^0$. Suppose that there exists a number b with $f(b) = -4e^b$. Since $e^b > 0$, we get $f(b) < 0$. On the other hand, $f(0) = 4 > 0$. The function f is continuous on $[0,b]$, so we can apply the intermediate value theorem with $\lambda = 0$: We can find a $\xi \in (0,b)$ with $f(\xi) = 0$. Then

$$0 = f(\xi) = \pm 4e^\xi \Longrightarrow e^\xi = 0.$$

But the function e^x is positive. So this is impossible. By contradiction, there is no such b with $f(b) = -4e^{-b}$. Therefore, the $+$ sign is correct for all $x \in [0,\infty)$:

$$f(x) = 4e^x.$$

6. (a) Let $n \geq 2$ and $b_1, b_2, \ldots, b_n$ be positive numbers with

$$b_1 \cdot b_2 \cdots b_n = 1.$$

Show that (using induction)

$$b_1 + b_2 + \cdots + b_n \geq n.$$

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For $n = 2$ we need to prove

$$b_1 b_2 = 1 \Longrightarrow b_1 + b_2 \geq 2.$$

We have

$$b_1 + b_2 = b_1 + \frac{1}{b_1} = \frac{b_1^2 + 1}{b_1} \geq 2 \Leftrightarrow b_1^2 + 1 \geq 2b_1 \Leftrightarrow b_1^2 - 2b_1 + 1 \geq 0 \Leftrightarrow (b_1 - 1)^2 \geq 0,$$

which is true.

Assume that $P(n)$ is true, i.e. if $b_1 b_2 \cdots b_n = 1$, then we have

$$b_1 + b_2 + \cdots + b_n \geq n.$$

We need to prove $P(n+1)$ i.e. if $b_1 b_2 \cdots b_n \cdot b_{n+1} = 1$, then we have

$$b_1 + b_2 + \cdots + b_n + b_{n+1} \geq n + 1.$$

If all the b_i's are 1, the result is obvious. If one b_i is not 1, then it has to be either > 1 or < 1. Then there must be another b_j with b_j either < 1 or > 1, respectively. Otherwise we have

$$b_i > 1, \quad b_j \geq 1, \forall j \neq i \Longrightarrow b_1 b_2 \cdots b_{n+1} > 1$$

or

$$b_i < 1, \quad b_j \leq 1, \forall j \neq i \Longrightarrow b_1 b_2 \cdots b_{n+1} < 1,$$

and this contradict $b_1 b_2 \cdots b_{n+1} = 1$.

By rearranging the b_i's, we can assume that $b_n > 1$ and $b_{n+1} < 1$. This implies

$$(b_n - 1)(1 - b_{n+1}) > 0 \Leftrightarrow b_n - 1 - b_n b_{n+1} + b_{n+1} > 0. \tag{4}$$

Since

$$b_1 b_2 \cdots (b_n b_{n+1}) = 1$$

by inductive hypothesis we get

$$b_1 + b_2 + \cdots + b_n b_{n+1} \geq n.$$

We add this to (4) to get

$$b_1 + b_2 + \cdots + b_n b_{n+1} + b_n - 1 - b_n b_{n+1} + b_{n+1} > n \Longrightarrow b_1 + b_2 + \cdots + b_n + b_{n+1} > n + 1.$$

This proves $P(n+1)$.

(ii) State and prove the Arithmetic Mean – Geometric Mean Inequality for n non-negative numbers $a_1, a_2, \ldots, a_n$.

AM–GM:

$$\text{GM} = \sqrt[n]{a_1 \cdot a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} = \text{AM}.$$

Naturally the geometric mean can also be written as $\text{GM} = (a_1 \cdot a_2 \cdots a_n)^{1/n}$.

Proof: If any of the a_i's are zero, then $\text{GM} = 0$, while $\text{AM} \geq 0$. This gives the result. If all a_i's are non zero, then $G = \text{GM} \neq 0$. We scale the a_i's by G. Define

$$b_i = \frac{a_i}{G}, \quad i = 1, \ldots, n.$$

We have

$$b_1 b_2 \cdots b_n = \frac{a_1 a_2 \cdots a_n}{G^n} = \frac{a_1 a_2 \cdots a_n}{a_1 a_2 \cdots a_n} = 1,$$

so that we can apply part (i). This gives

$$b_1 + b_2 + \cdots + b_n \geq n \Leftrightarrow \frac{a_1}{G} + \frac{a_2}{G} + \cdots + \frac{a_n}{G} \geq n \Leftrightarrow \frac{a_1 + a_2 + \cdots + a_n}{G} \geq n \Leftrightarrow \frac{a_1 + a_2 + \cdots a_n}{n} \geq G,$$

and this is exactly

$$\text{AM} \geq \text{GM}.$$

(b) Define the sequence $\langle x_n \rangle$ by

$$x_1 = 1, \quad x_{n+1} = \frac{1}{1 + x_n}, \quad n \in \mathbb{N}.$$

Use induction to prove that

$$x_{2n} < x_{2n+2} < x_{2n+1} < x_{2n-1}, \quad n \in \mathbb{N}.$$

Deduce that the subsequences of even and odd subscripts converge to the same limit. What does this imply for the sequence $\langle x_n \rangle$?

Let

$$P(n) : x_{2n} < x_{2n+2} < x_{2n+1} < x_{2n-1}, \quad n \in \mathbb{N}.$$

We compute

$$x_2 = 1/(1+1) = 1/2, \quad x_3 = \frac{1}{1 + 1/2} = \frac{2}{3}, \quad x_4 = \frac{1}{1 + 2/3} = \frac{3}{5}.$$

We see that

$$\frac{1}{2} < \frac{3}{5} < \frac{2}{3} < 1$$

i.e.

$$x_2 < x_4 < x_3 < x_1.$$

This means that $P(1)$ is true.

We assume that $P(n)$ is true and we prove that $P(n+1)$ is true. This means we assume that

$$P(n): \quad x_{2n} < x_{2n+2} < x_{2n+1} < x_{2n-1}$$

is true and we need to prove that

$$P(n+1): \quad x_{2n+2} < x_{2n+4} < x_{2n+3} < x_{2n+1}.$$

We have

$$1+x_{2n} < 1+x_{2n+2} < 1+x_{2n+1} < 1+x_{2n-1} \Longrightarrow \frac{1}{1+x_{2n}} > \frac{1}{1+x_{2n+2}} > \frac{1}{1+x_{2n+1}} > \frac{1}{1+x_{2n-1}}$$

$$\Longrightarrow x_{2n+1} > x_{2n+3} > x_{2n+2} > x_{2n}. \qquad (5)$$

This only proves $x_{2n+1} > x_{2n+3}$ from $P(n+1)$. We use the inequality at the left end of (5) to deduce from this that

$$1+x_{2n+3} < 1+x_{2n+1} \Longrightarrow \frac{1}{1+x_{2n+3}} > \frac{1}{1+x_{2n+1}} \Longrightarrow x_{2n+4} > x_{2n+2}.$$

We use the middle inequality in (5):

$$x_{2n+2} < x_{2n+3} \Longrightarrow 1+x_{2n+2} < 1+x_{2n+3} \Longrightarrow \frac{1}{1+x_{2n+2}} > \frac{1}{1+x_{2n+3}} \Leftrightarrow x_{2n+3} > x_{2n+4}.$$

This proves $P(n+1)$.

The subsequence of odd subscripts is strictly decreasing and bounded below by x_2, so it converges to its largest lower bound. The subsequence of even subscripts is strictly increasing and bounded above by x_1, so it converges to its supremum. Say

$$\lim_n x_{2n} = l, \quad \lim_n x_{2n-1} = m.$$

Since $\langle x_{2n+1} \rangle$ is a subsequence of $\langle x_{2n-1} \rangle$, we also have $\lim x_{2n+1} = m$. We need to prove that $l = m$. We look at

$$x_{2n+1} = \frac{1}{1+x_{2n}}$$

and take limits of both sides. We get

$$m = \lim x_{2n+1} = \frac{1}{1+l} \Longrightarrow m + ml = 1.$$

Similarly, we look at

$$x_{2n} = \frac{1}{1+x_{2n-1}}$$

and take limits of both sides. We get

$$l = \lim x_{2n} = \frac{1}{1+m} \Longrightarrow l + ml = 1.$$

We subtract to get $l = m$. As both subsequences of even and odd subscripts converge to the same limit, the whole sequence converges to this limit.